GPS and Relativity: How Einstein Helps Us Find Our Way Home

QuarkNet Workshop

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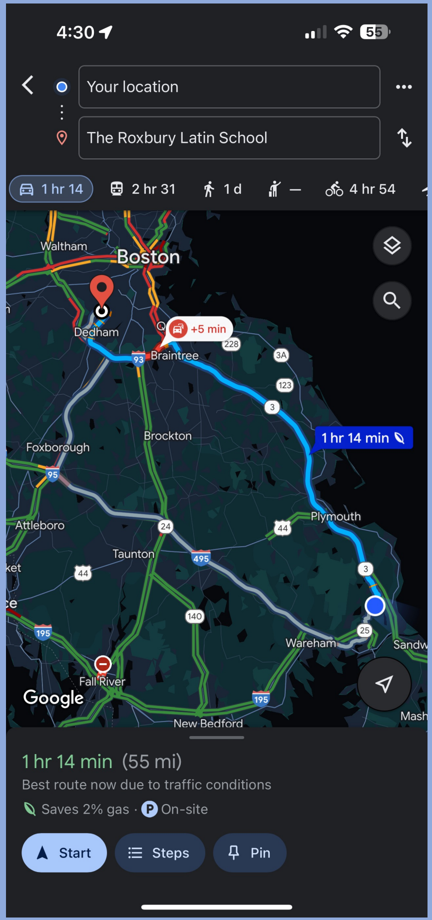
I often use a navigation app to estimate travel time to a destination.



My phone gets warm when I use a navigation app to track my progress.

Why?

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Why?

How does a navigation app determine my phone location?

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Why?

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What is GPS?

Detour: Location, Location, Location

Street Address:	1 number locates a point on a line
Longitude (λ), Latitude (ϕ):	2 numbers locate a point on a surface
λ, ϕ, Altitude:	3 numbers locate a point in space

Numbers are measured from a reference frame coordinate origin and set of axes. For example, an Earth reference frame:
origin at Earth's center, one axis through the crossing point of Prime Meridian and equator, one axis through the geographic N and S poles,
λ from the Greenwich Prime Meridian (in the equatorial plane),
ϕ from the equator (North or South along a meridian), and
Altitude above of below mean sea level (perpendicular to the geoid).

Celestial Reference Frame

On 1 January, 1998, the International Astronomical Union established the non-rotating International Celestial Reference System, since realized by three successive International Celestial Reference Frames.

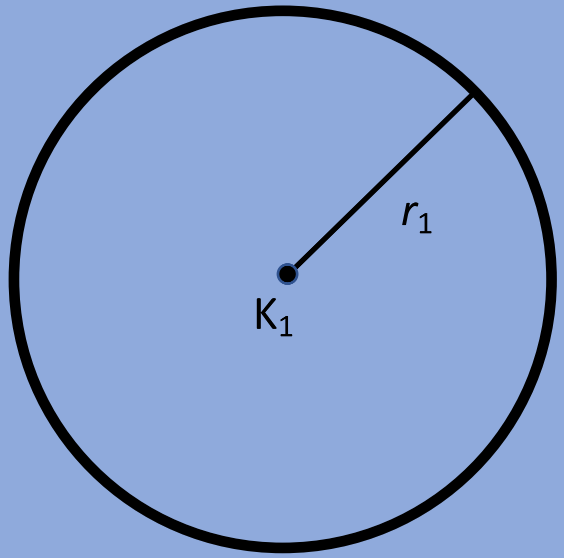
The barycenter (center of mass) of the Solar System is fixed with reference to many distant radio sources (mostly quasars) measured with milliarcsecond precision and corresponding optical sources measured by the Gaia satellite.

The most recent (1 January 2022) frame for specifying the position and motion of objects relative to the Solar System barycenter is specified by ICRF3 and Gaia-CRF3.

From the position of an object relative to Earth and the position of Earth relative to Gaia-CFR3, one can specify the position of the object relative to the stars.

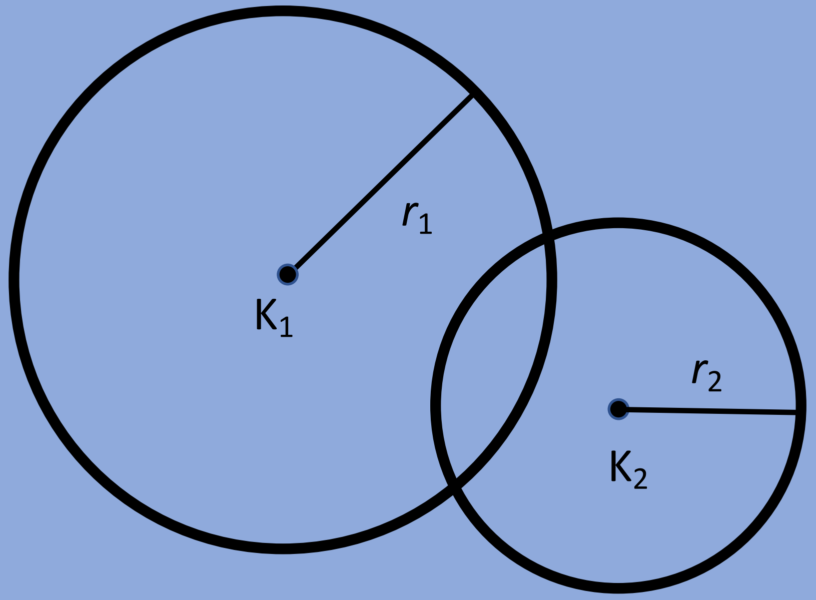
Locate an Unknown Point (1)

On a plane, if the distance from a known location (K_1) to an unknown point (P) is r_1, then P must be on a circle of radius r_1 around K_1.



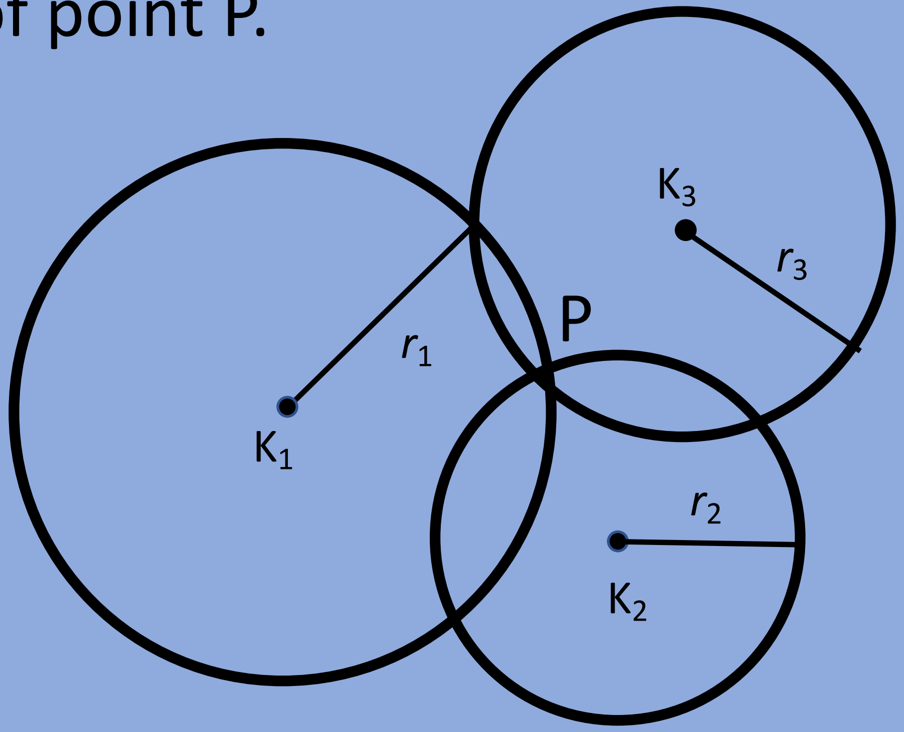
Locate an Unknown Point (2)

Specifying a second known distance (r_2) from a known location (K_2) limits point P to one of two intersections.



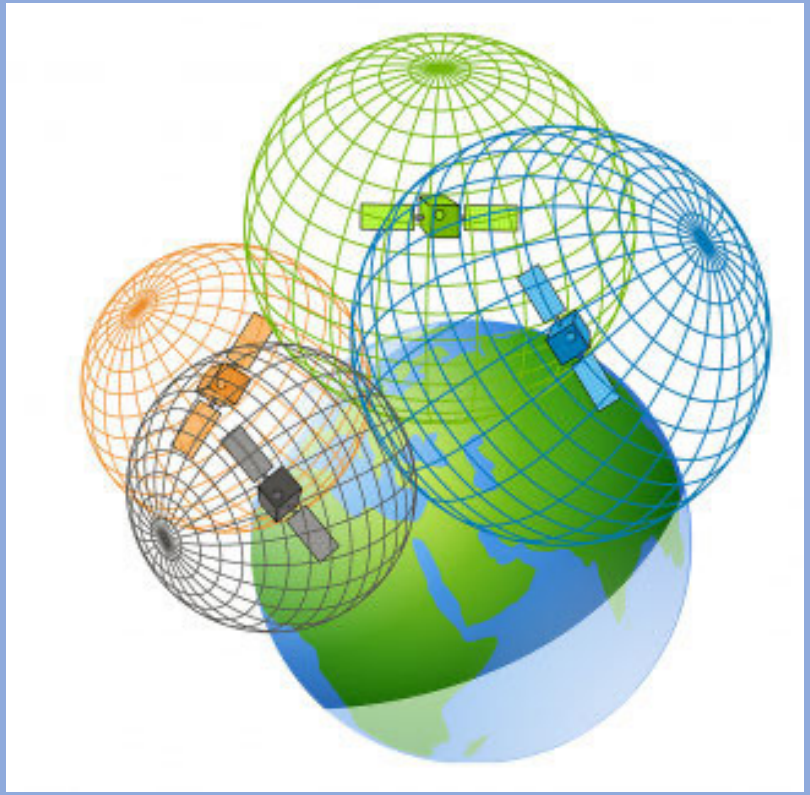
Locate an Unknown Point (3)

A third circle with known radius (r_3) from a known location (K_3) identifies the location of point P.



Locate an Unknown Point (4)

In three dimensions, we replace the circles with spheres.



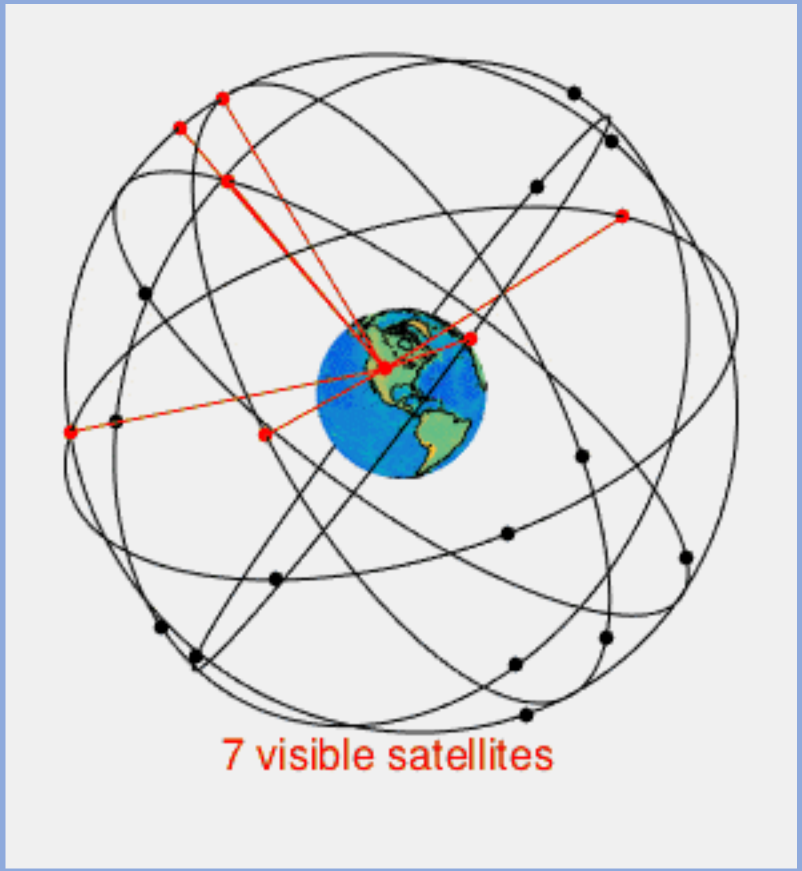
Back to GPS: Global Positioning System

GPS (descended from the U. S. Navy's 1970s NAVSTAR system) is a constellation of at least 24 satellites (now 31, with spares).

Orbits are inclined 55 deg to Earth's equator at 4.2 Earth radii (26,580 km) from Earth center.

Orbital periods are 12 hours.

(Image from Wikipedia "GPS")



GPS Hardware and App Functions (1)

(1) Receive signals from at least 4 GPS satellites

(2) Decode signals to read satellite locations

(3) Calculate satellite distances

(4) Calculate receiver position and clock offset

(5) Display the calculated position on a local map

GPS Hardware and App Functions (2)

(1) Receive signals from at least 4 GPS satellites
[We leave this to electronics engineers.]

(2) Decode signals to read satellite locations

(3) Calculate satellite distances

(4) Calculate receiver position and clock offset

(5) Display the calculated position on a local map

GPS Hardware and App Functions (3)

(1) Receive signals from at least 4 GPS satellites
[We leave this to electronics engineers.]

(2) Decode signals to read satellite locations
[GPS satellites broadcast their positions, clock times, and ephemeris parameters. Decoding done thanks to software engineers.]

(3) Calculate satellite distances

(4) Calculate receiver position and clock offset

(5) Display the calculated position on a local map

GPS Hardware and App Functions (4)

(1) Receive signals from at least 4 GPS satellites
[We leave this to electronics engineers.]

(2) Decode signals to read satellite locations
[GPS satellites broadcast their positions, clock times, and ephemeris parameters. Decoding done thanks to software engineers.]

(3) Calculate satellite distances

(4) Calculate receiver position and clock offset
[Continuous calculation for this math problem makes our phones warm.]

(5) Display the calculated position on a local map

GPS Hardware and App Functions (5)

(1) Receive signals from at least 4 GPS satellites
[We leave this to electronics engineers.]

(2) Decode signals to read satellite locations
[GPS satellites broadcast their positions, clock times, and ephemeris parameters. Decoding done thanks to software engineers.]

(3) Calculate satellite distances

(4) Calculate receiver position and clock offset
[This math problem makes our phones warm.]

(5) Display the calculated position on a local map
[This we leave to our navigation app software.]

Receiver-to-Satellite Distances (1)

(1) How do our navigation apps calculate receiver to GPS satellite distances?

Receiver-to-Satellite Distances (2)

(1) How do our navigation apps calculate receiver to GPS satellite distances?

$$d = c(t_{reciever} - t_{satellite}), \qquad c = 299{,}792{,}458 \text{ m/s in vacuum}$$

Receiver-to-Satellite Distances (3)

(1) How do our navigation apps calculate receiver to GPS satellite distances?

$$d = c(t_{receiver} - t_{satellite}), \qquad c = 299{,}792{,}458 \text{ m/s in vacuum}$$

(2) What effects must navigation apps account for to calculate accurate distances (+/- 10 m)?

Receiver-to-Satellite Distances (4)

(1) How do our navigation apps calculate receiver to GPS satellite distances?

$$d = c(t_{receiver} - t_{satellite}), \qquad c = 299{,}792{,}458 \text{ m/s in vacuum}$$

(2) What effects must navigation apps account for to calculate accurate distances (+/- 10 m)?

(a) Receiver clocks are not as precise as GPS satellite atomic clocks. Fourth satellite signal is needed to calculate receiver clock time offset and 3-d receiver position, *i. e*. 4 signals for 4 unknowns.

Receiver-to-Satellite Distances (5)

(1) How do our navigation apps calculate receiver to GPS satellite distances?

$$d = c(t_{\text{receiver}} - t_{\text{satellite}}), \qquad c = 299{,}792{,}458 \text{ m/s in vacuum}$$

(2) What effects must navigation apps account for to calculate accurate distances (+/- 10 m)?

(a) Receiver clocks are not as precise as GPS satellite atomic clocks. Fourth satellite signal is needed to calculate receiver clock time offset and 3-d receiver position, *i. e*. 4 signals for 4 unknowns.

(b) Ionosphere time delay can be corrected with two signal frequencies. Atmosphere time delay can be modeled with atmosphere model.

Receiver-to-Satellite Distances (6)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

Question: Do GPS satellite clocks tick slower or faster than Earth clocks due to satellite motion relative to Earth?

Receiver-to-Satellite Distances (7)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

Question: Do GPS satellite clocks tick slower or faster than Earth clocks due to satellite motion relative to Earth?

Answer: Time dilation for the moving GPS clocks yields a slower tick rate for GPS clocks than Earth clocks.

Receiver-to-Satellite Distances (8)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

Exercise: Calculate the orbital speed of GPS satellites in km/s and v/c.

(r_{GPS} = 26,580 km, T_{GPS} = 12 hr)

Receiver-to-Satellite Distances (9)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

Exercise: Calculate the orbital speed of GPS satellites in km/s and *v*/c.

(r_{GPS} = 26,580 km, T_{GPS} = 12 hr)

Answer: $v_{GPS} = \frac{2\pi(26580\ \mathrm{km})}{43200\ s} = 3870\ \mathrm{m/s}$ $v_{GPS}/c = 1.29 \times 10^{-5}$

Receiver-to-Satellite Distances (10)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

Exercise: Calculate the ratio g_{GPS}/g_{Earth}.

[r_{Earth} = 6380 km, altitude$_{GPS}$ = 20200 km, r_{GPS} = 26580 km = 4.17 r_{Earth}]

Receiver-to-Satellite Distances (11)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

Exercise: Calculate the ratio g_{GPS}/g_{Earth}.

Answer: $g_{GPS}/g_{Earth} = (r_{Earth}/r_{GPS})^2 = (6380\text{ km}/26580\text{ km})^2 = (0.24)^2 = 0.0576$
$= 1/17.4$

Receiver-to-Satellite Distances (12)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

Question: Do GPS satellite clocks tick slower or faster than Earth clocks due to lower gravity field at GPS altitude than on Earth?

Receiver-to-Satellite Distances (13)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

Question: Do GPS satellite clocks tick slower or faster than Earth clocks due to lower gravity field at GPS altitude than on Earth?

Answer: GPS clocks tick faster than Earth clocks due to their lower gravity field .

Receiver-to-Satellite Distances (14)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

Question: How significant are the relativity effects?

Question: Which of the two relativity effects is larger?

Receiver-to-Satellite Distances (15)

But wait, there's more:

Corrections are made to GPS satellite clock frequencies due to:
(1) orbital motion relative to Earth (Special Relativity)

(2) lower ambient gravity field in orbit (General Relativity).

If relativity corrections were not applied to the GPS satellite clocks, the error rate for navigation apps would be 7.3 miles/day = 11.7 km/day.

Combined Relativity Effects Are Large

Special Relativity effect > General Relativity effect?

Special Relativity effect < General Relativity effect?

TBD - after a BREAK!

Einstein did not invent relativity!

Galileo Galilei (1564 – 1642)

Galilean Relativity

Galileo is generally acknowledged to be the first to propose the relativity principle. As he says in his “Note to the Discerning Reader” in his *Dialogue the Two Chief World Systems*,

I shall try to show that all experiments practicable upon the earth are insufficient measures for proving its mobility, since they are indifferently adaptable to an earth in motion or at rest.

[Such experiments do not prove that Earth is at rest. Earth could be moving.]

(G. Galilei, *Dialogue Concerning the Two Chief World Systems*, S. Drake, trans., University of California Press, Berkley, 1962, p. 6.)

Galilean Relativity – Galileo's Thought Experiment

"Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals.

Galilean Relativity – Galileo's Thought Experiment

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. **Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it.**

Galilean Relativity – Galileo's Thought Experiment

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. **With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions, the drops fall into the vessel beneath;**

Galilean Relativity – Galileo’s Thought Experiment

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions, the drops fall into the vessel beneath; **and in throwing something to your friend, you need throw it no more strongly in one direction than in another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.**

Galilean Relativity – Galileo's Thought Experiment

When you have observed all these things carefully…, have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. **You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still."** (Emphasis added)

(G. Galilei, *Dialogue Concerning the Two Chief World Systems*, S. Drake, trans., University of California Press, Berkley, 1962, p. 186-187.)

Public Experimental Test in 1641

From the official record of the demonstration:

Mr. Gassendi, always having been curious to seek to justify by experiments the truth of the speculations proposed to him by philosophy and finding himself in Marseilles with his Lordship the Count of Allais in the year 1641, **demonstrated, on a galley which set out to sea** designedly by order of this Prince,... **that a stone dropped from the very top of the mast, while the galley is sailing with all force and speed possible, will not fall in any other spot than it would if this same galley were stopped and immobile**.

(Emphasis added)

(R. Dugas, *Mechanics in the Seventeenth Century*, F. Jacquot, trans., Éditions du Griffon, Neuchatel, 1958, p. 110. Quoted in A. Koyre, *Metaphysics and Measurement*, Harvard University Press., Cambridge, 1968, pp. 126-127.)

Galilean Relativity

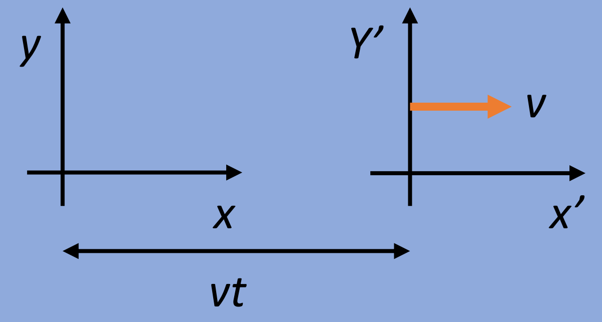
Mechanical experiments, like like Galileo's proposed thought experiments and Gassendi's falling stone, cannot distinguish between reference frames at rest and reference frames moving with constant velocity relative to a frame at rest.

Galileo argued that Earth could be moving, contrary to Aristotle, without our being able to recognize it. He guessed that the effects of circular motion would be too small to notice.

Galilean Transformations

Galilean transformations relate measurements of event coordinates (time and position) in two inertial reference frames moving at constant velocity relative to each other. Time is assumed to be universal. Origins and axes overlap at time $t = t' = 0$, after which $t = t'$.

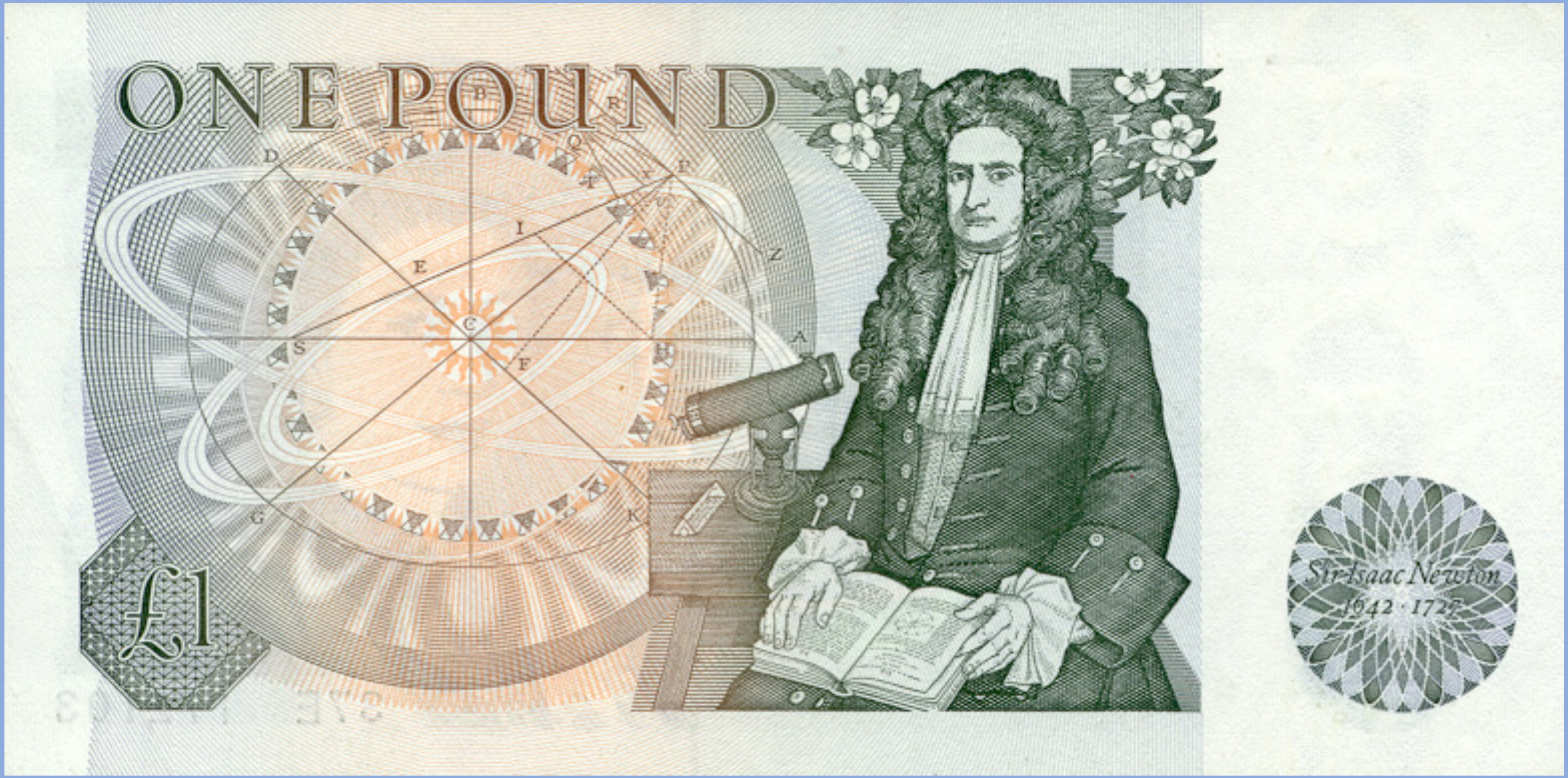
The Rocket system (t', x', y', z') moves with constant velocity v in the +x-direction of the Lab system (t, x, y, z).



Galilean Transformations:

$t = t'$, $x = x' + vt'$, $y = y'$, $z = z'$

Isaac Newton (1642 – 1727)



Newtonian Relativity In Mechanics

Absolute, true, and mathematical time … flows equably without relation to anything external.

Absolute space ... without relation to anything external remains always similar [homogeneous] and immovable.

(I. Newton, *Mathematical Principles of Natural Philosophy*, A. Motte, trans., revised by F. Cajori, University of California Press, Berkeley, 1962, p. 6.)

Newton acknowledged that absolute time and space are not observable. We make measurements of what he termed "relative, apparent, or common time" compared to reference clocks and "relative space" compared to observable objects.

Newtonian Relativity In Mechanics

Reference frames in which Newton's inertia law (Law I) holds are called **inertial reference frames**.

Any reference frame moving with constant vector velocity relative to an inertial reference frame is also an inertial frame.

Or, as Newton stated in Corollary V of his Laws of Motion:

"The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves forwards in a right line without any circular motion." (I. Newton, *ibid*. p.20.)

Newtonian Relativity In Mechanics

Newton extended this idea in Corollary VI to include frames in which all bodies are moved with equal acceleration in straight lines, as in gravitational free fall approximated in the International Space Station.

How inertial are we?

- Earth rotation: T_E = 86164 s (sidereal day)
  Time is to change direction of motion by 1 degree due to Earth spin = ?

- Earth orbit around Sun: $T_{Eorb} = 3.16 \times 10^7$ s = 365.243 days = 1 yr
  Time is to change direction of motion by 1 degree due to Earth orbit = ?

- Sun orbit around Galaxy: $T_{Sorb} = 7.14 \times 10^{15}$ s = 2.26×10^8 yr
  Time is to change direction of motion by 1 degree due to solar system Galactic orbit = ?

Calculate one of these approximate times.

How inertial are we?

- Earth rotation: T_E = 86164 s (sidereal day)
  Time is to change direction of motion by 1 degree due to Earth spin = 4 minutes
- Earth orbit around Sun: $T_{Eorb} = 3.16 \times 10^7$ s = 365.243 days = 1 yr
  Time is to change direction of motion by 1 degree due to Earth orbit = 1 day
- Sun orbit around Galaxy: $T_{Sorb} = 7.14 \times 10^{15}$ s = 2.26×10^8 yr
  Time is to change direction of motion by 1 degree due to solar system Galactic orbit = 6.3×10^5 yr

How inertial are we?

- Earth rotation: $r_E = 6.38 \times 10^6$ m $T_E = 86164$ s (sidereal day)
  $v_0 = 465$ m/s at equator
  $a_0 = ?$
- Earth orbit around Sun: $r_{Eorb} = 1.5 \times 10^{11}$ m = 1 AU $T_{Eorb} = 3.16 \times 10^7$ s = 1 yr
  $v_{Eorb} = 29.8$ km/s = 3×10^4 m/s
  $a_{Eorb} = ?$
- Sun orbit around Galaxy: $r_{Sorb} = 2.53 \times 10^{20}$ m = 2.67×10^4 ly
  $T_{Sorb} = 7.14 \times 10^{15}$ s = 2.26×10^8 yr
  $v_{Sorb} = 223$ km/s = 2.23×10^5 m/s
  $a_{Sorb} = ?$

Calculate one of these centripetal acceleration values.

How inertial are we?

- Earth rotation:
  $r_E = 6.38 \times 10^6$ m $T_E = 86164$ s (sidereal day)
  $v_0 = 465$ m/s at equator
  $a_0 = 0.034$ m/s^2
- Earth orbit around Sun:
  $r_{Eorb} = 1.5 \times 10^{11}$ m = 1 AU $T_{Eorb} = 3.16 \times 10^7$ s = 1 yr
  $v_{Eorb} = 29.8$ km/s = 3×10^4 m/s
  $a_{Eorb} = 0.006$ m/s^2 = 6×10^{-3} m/s^2
- Sun orbit around Galaxy:
  $r_{Sorb} = 2.53 \times 10^{20}$ m = 2.67×10^4 ly
  $T_{Sorb} = 7.14 \times 10^{15}$ s = 2.26×10^8 yr
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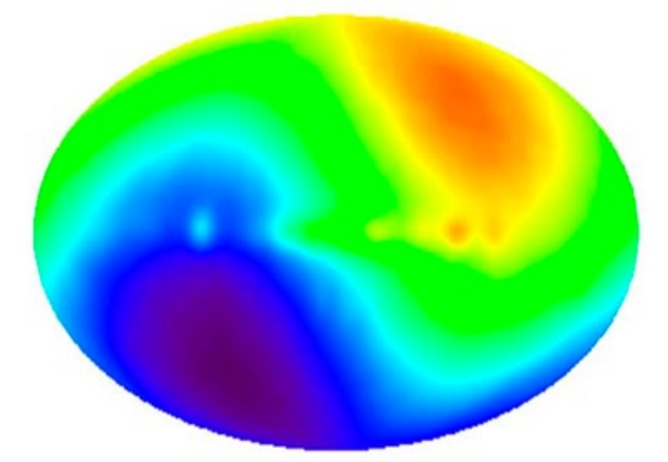
  $a_{Sorb} = 2.0 \times 10^{-10}$ m/s^2

CMB Reference Frame

A Planck satellite research group has derived the speed and direction of the solar system relative to the Cosmic Microwave Background (CMB) from the dipole term in the CMB radiation distribution over space. The CMB is the closest approximation we have to a "stationary" reference frame.



The Galactic equator stretches left to right through the plot center.

Red and orange indicate slightly (few parts per thousand) warmer regions. Violet and blue indicate slightly cooler regions.

The Sun's 370 km/s motion through the CMB is toward the constellation Crater near the border of Leo and Virgo in the upper right quadrant of the plot. Solar system motion through the CMB is nearly opposite in direction to solar motion around the Galaxy.

A 19th Century Question from Optics

Could the luminiferous (*i. e.* light-bearing) aether serve as a stationary reference frame in absolute space?

(Note: “Aether”is often spelled “ether.” I use “aether” to correspond to the original Greek spelling and to avoid confusion with chemical ethers, *e. g.* $CH_3CH_2OCH_2CH_3$.)

In Greek mythology, Aether was the son of Erebus (darkness) and Nyx (night). He was the personification of the pure essence breathed by the gods.

For Aristotle, aether was the fifth element (translated as “quintessence” in Latin) that constituted the heavens. According to Aristotle, the natural circular motion of the aether carried the heavenly bodies in their motions around Earth.

Luminiferous Aether as a Mechanical Medium

Christiaan Huygens (1629-1695)

Amsterdam Shop Wall

Huygens’ Aether (1)

In *Treatise on Light* (1690), Christiaan Huygens emphasized the analogy between light and sound:

“[W]hen one considers the extreme speed with which light spreads on every side, and how, when it comes from different regions, even those directly opposite, the rays traverse one another without hindrance,

Huygens’ Aether (2)

In his *Treatise on Light* (1690), Christiaan Huygens emphasized the analogy between light and sound:

“[W]hen one considers the extreme speed with which light spreads on every side, and how, when it comes from different regions, even those directly opposite, the rays traverse one another without hindrance, **one may well understand that when we see a luminous object, it cannot be my any transport of matter coming to us from this object. . . .**

Huygens’ Aether (3)

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It will follow that this movement, impressed on the intervening matter, is successive; and consequently it spreads, as Sound does, by spherical surfaces and waves”
(C. Huygens, *Treatise on Light*, S. Thompson, trans., Dover Publications, New York, 1962, pp. 3-4.)

Huygens' Aether (4)

"[T]his matter. . .in which the movement coming from the luminous body is propagated, which I call Ethereal matter, . . . is not the same that serves for the propagation of Sound."
(C. Huygens, *Treatise on Light*, p. 11.)

Huygens conceived light vibrations as longitudinal vibrations due to elastic particle collisions in analogy to sound vibrations in air, but he noted that the aethereal matter must exist in vacuum spaces at the tops of barometers and in the region between Earth, Sun, and stars.

Newton's Aether

Huygens' wave theory of light propagation was neglected in the 1700s in favor of Newton's suggestion in Query 29 of his *Opticks*: "Are not the Rays of Light very small Bodies emitted from shining Substances?"

(I. Newton, *Opticks*, 4th ed., 1730, Dover Publications, New York, 1952, p. 370.)

Newton, however, did imagine an aethereal medium that influences light particles and transmits heat vibrations through vacuum (Query 18), produces refraction by changes in density (Query 19), produces diffraction (Query 20), causes gravitational attraction (Query 21), but it too thin to inhibit celestial motions (Query 22).

(*Ibid*. pp. 352-353.)

Luminiferous Aether (1)

With the revival of the wave model of light by Thomas Young (1803) to explain double-source interference phenomena and by Augustin-Jean Fresnel (1815-1818) to explain diffraction, the idea of an aether medium to carry light vibrations was revived.

Luminiferous Aether (2)

James Clerk Maxwell (1864) developed a theory of electromagnetic (EM) vibrations in the aether in which the speed EM wave speed equaled the speed of light: $c = \sqrt{{}^{2\mathrm{k_C}}/_{\mathrm{k_A}}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cong 3\mathrm{x}10^8\ \mathrm{m/s}.$

k_C = 9 x 10^9 N m^2/C^2 is the constant in Coulomb's Law,
k_A = 2 x 10^7 N/A^2 is the constant in Ampere's Law.

Maxwell identified the EM waves as vibrations electric and magnetic fields in an aether at rest.

Heinrich Hertz (1887-1888) demonstrated the existence of Maxwell's electromagnetic waves and confirmed they traveled at speed of light.

Problems with a Mechanical Aether (1)

A mechanical aether must fill astronomical space, which suggested that it was an expansive fluid.

Problems with a Mechanical Aether (2)

A mechanical aether must fill astronomical space, which suggested that it was an expansive fluid.
But fluids support longitudinal waves, not transverse waves (except on surfaces, *e. g*. water waves). Light waves, as indicated by polarization phenomena, are transverse waves.

Solids, with strong interactions between particles, support transverse waves.
But how could the aether be solid?

Problems with a Mechanical Aether (3)

A mechanical aether must be highly rigid to support the extreme speed of light.

Problems with a Mechanical Aether (4)

A mechanical aether must be highly rigid to support the extreme speed of light.

<u>But</u> aether must have nearly zero resistance to the passage of matter and nearly zero viscosity to account for the apparently unimpeded motions of moons, planets, stars.

Problems with a Mechanical Aether (5)

Aether must be stationary relative to the Sun with Earth moving through it to account for Bradley's observation of stellar aberration.

Problems with a Mechanical Aether (6)

Aether must be stationary relative to the Sun with Earth moving through it to account for Bradley's observation of stellar aberration.

But aether must be stationary relative to Earth to account for the null result of Michelson-Morley experiment.

These results are contradictory!

How were they obtained?

Stellar Aberration (1)

Vertical Telescope fixed to a Chimney

In 1725-1728, astronomer James Bradley and Samuel Molyneux observed γ Draconis, a bright star near the ecliptic pole that passed nearly overhead in London, to search for stellar parallax, which they did not find. Bradley, however, discovered aberration, the apparent change in a star's direction due to Earth's motion.

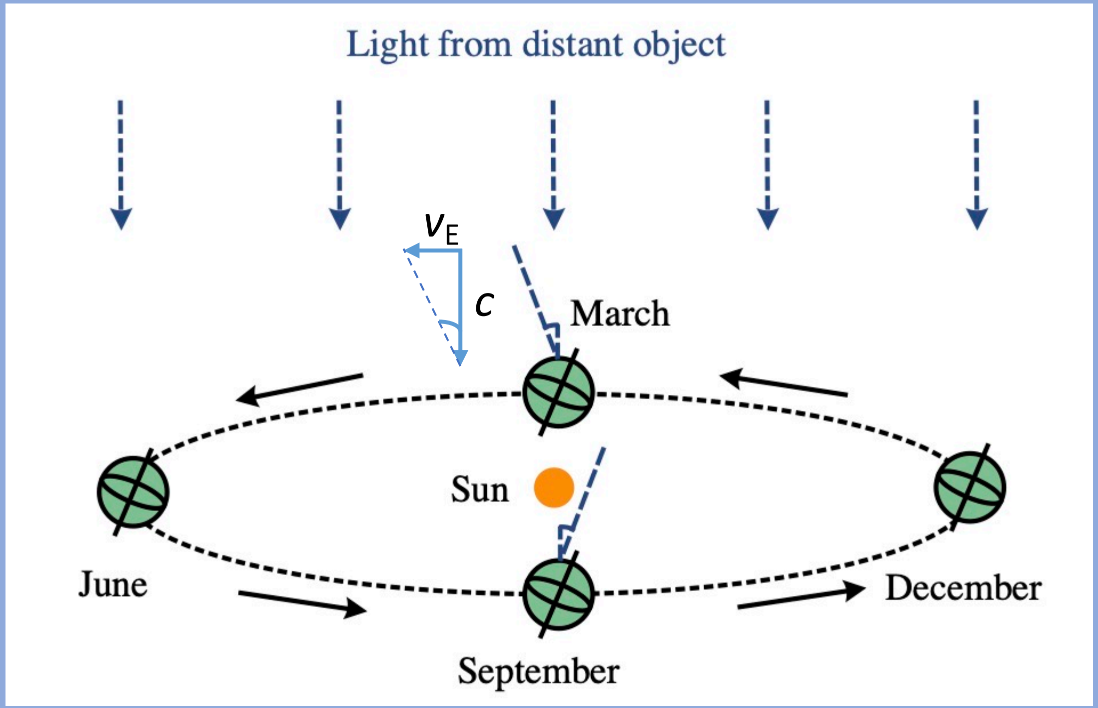
James Bradley (1692-1762)

Stellar Aberration (2)

Bradley found that the shift in the apparent north-south position of γ Draconis (and near-by stars) was greatest in March and September. A parallax shift would have been greatest in June and December. The maximum shift of 20.2" indicated that

$c = v_{\text{Earth}}/\tan 20.2'' = 3.06 \times 10^8$ m/s,

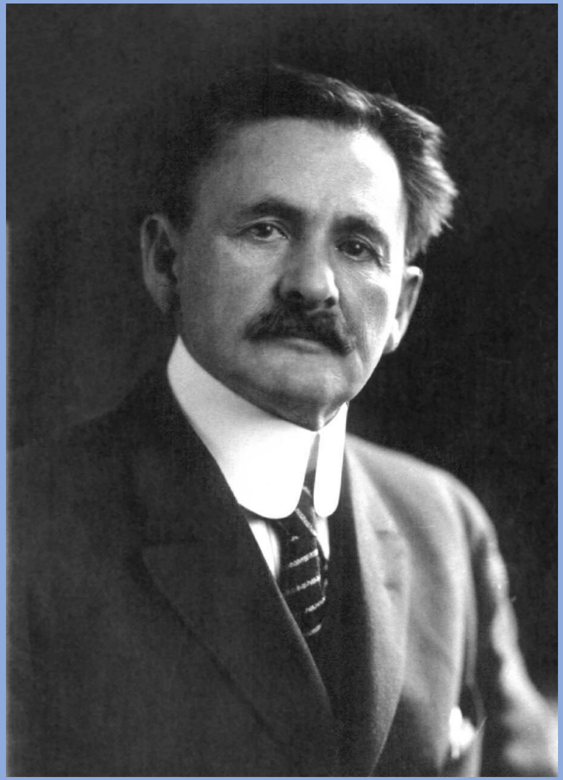
and indicated, in the 19th century wave theory, that Earth traveled through the aether.



Michelson-Morley Experiment (1)

Albert Michelson (1852-1931)

Nobel Prize in Physics (1907)



Edward Morley (1838-1923)

AAAS President (1895)

Michelson-Morley Experiment (2)

Theory with Mechanical Analogy

Airplanes A and B with airspeeds *c*, start at point O at the same time. They fly perpendicular routes of length L to points A and B, respectively, turn around quickly, and return to O.

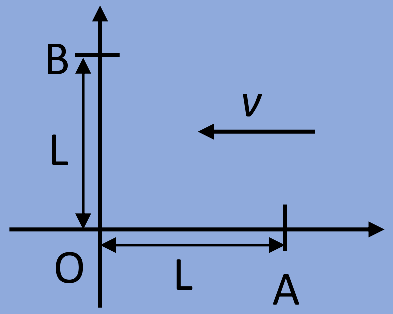
Wind (speed = *v*) blows in direction from A to O.

Which plane returns to point O first?

t_{OAO} = ?

t_{OBO} = ?

Take time to work this out.



Michelson-Morley Experiment (3)

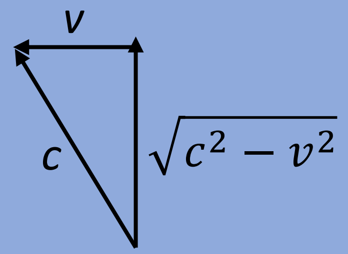
Theory with Wind

Round trip time for both planes in still air (v = 0): $t_0 = \frac{2L}{c}$

With wind blowing ($c > v$):

$$t_{\text{OAO}} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{L((c+v)+L(c-v)}{(c-v)(c+v)} = \frac{2Lc}{(c^2-v^2)} = \left(\frac{2L}{c}\right)\left(\frac{1}{1-\left(\frac{v}{c}\right)^2}\right)$$

$$t_{\text{OBO}} = \frac{L}{\sqrt{(c^2-v^2)}} + \frac{L}{\sqrt{(c^2-v^2)}} = \frac{2L}{\sqrt{(c^2-v^2)}} = \left(\frac{2L}{c}\right)\left(\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right)$$



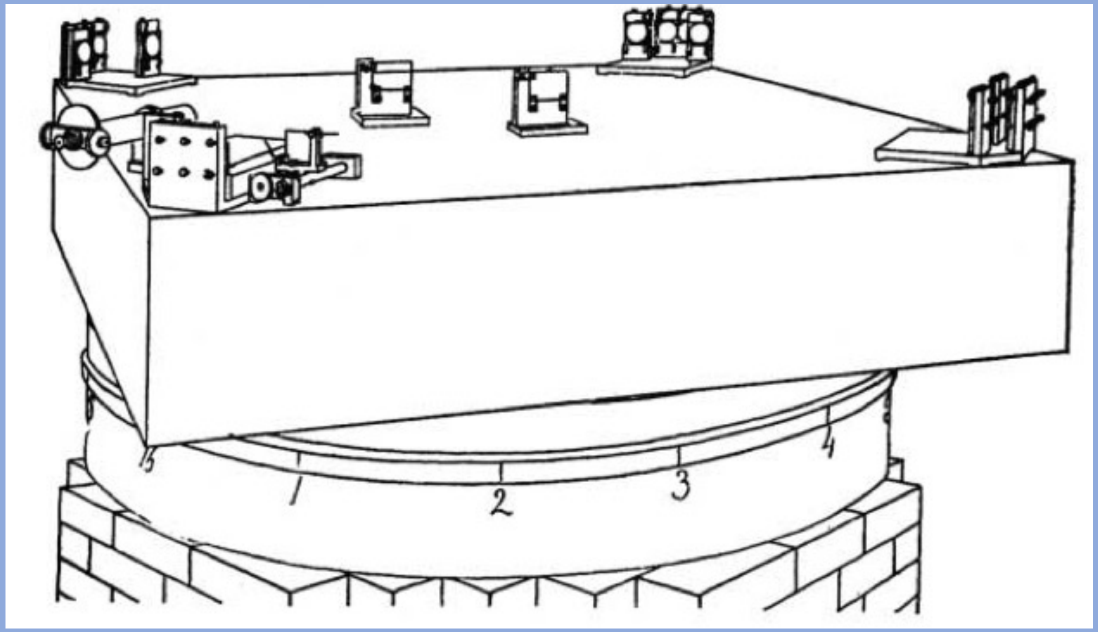
So, $t_{\text{OBO}} < t_{\text{OAO}}$, and airplane B returns to O first.

Michelson-Morley Experiment (4)

Albert Michelson hoped to detect Earth’s motion through the aether with an interferometer that compared the travel times of two perpendicular light beams. The beams were sent on perpendicular paths and recombined to produce a pattern of interference fringes (pattern of vertical bright and dark regions). Not knowing the direction of the presumed aether wind produced by Earth’s orbital (and rotational) motion, Michelson rotated his interferometer and looked for a shift in the fringe pattern.

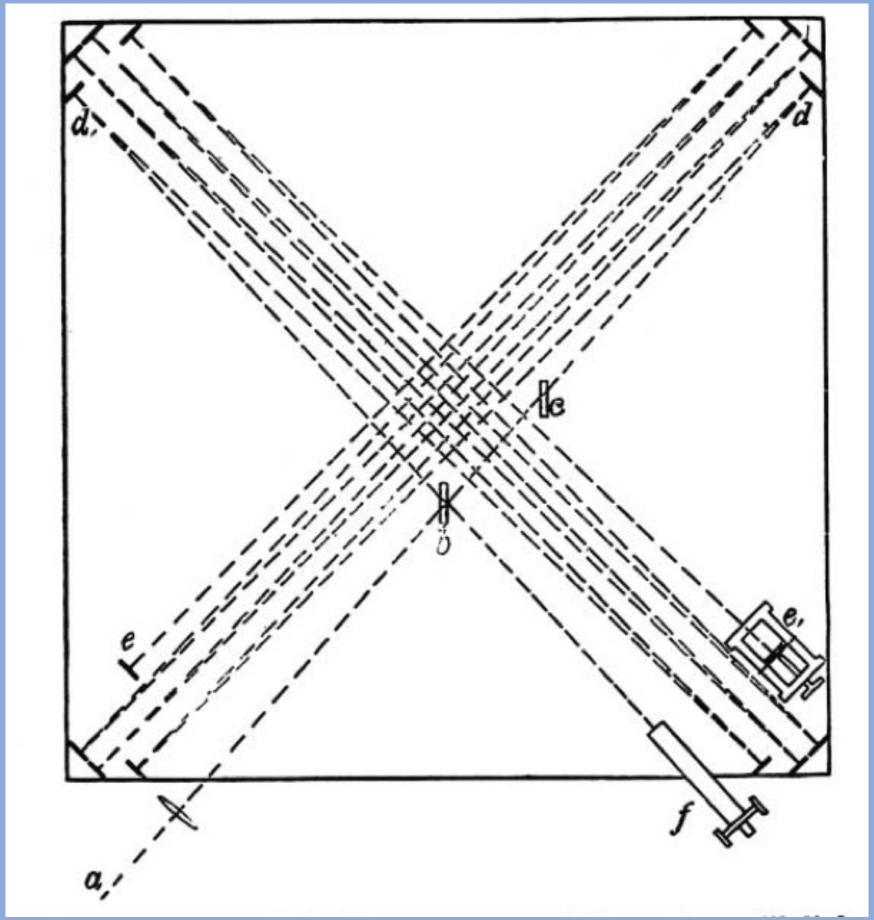
Michelson-Morley Experiment (5)

In 1887 Michelson built a refined version of the interferometer he had developed in Germany. Optical elements and viewing telescope were mounted on a stone block 0.3 m thick and 1.5 m square resting on a wooden ring that floated on mercury contained in an iron collar. This ensured minimal vibration and allowed smooth rotation.



Michelson-Morley Experiment (6)

With the light beams reflecting from several mirrors, the light path length (L) in each of the perpendicular arm was ~ 11 m. When brought together in the viewing telescope, the two beams produced interference fringes of vertical light and dark bands. The apparatus was slowly rotated to observe the expected 0.4 fringe shift.



Michelson-Morley Experiment (7)

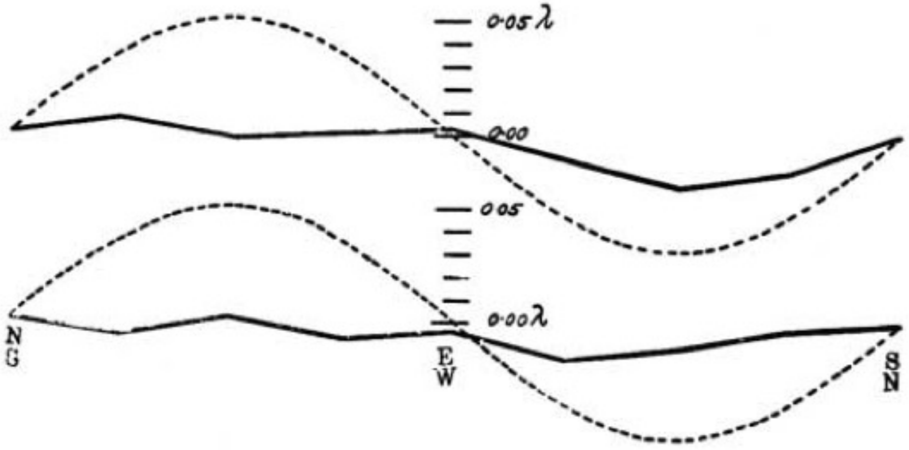
Much to Michelson’s surprise, there was no apparent fringe shift.
(Figures and text from *Am J. Sci.* **203**, **XXXIV**, 1887, pp.333-345.)

Results were incompatible with the existence of and aethereal medium for light.

As late as 1927, Michelson wondered, “without a medium [aether] how can the propagation of light waves be explained?”
(*Studies in Optics*, p. 161.)

The results of the observations are expressed graphically in fig. 6. The upper is the curve for the observations at noon, and the lower that for the evening observations. The dotted curves represent *one-eighth* of the theoretical displacements. It seems fair to conclude from the figure that if there is any dis-

6.



placement due to the relative motion of the earth and the luminiferous ether, this cannot be much greater than 0·01 of the distance between the fringes.

Initial Efforts at Reconciliation (1)

George FitzGerald (1889) proposed (*ad hoc*) that objects might shrink in the direction of motion through the aether by the appropriate ratio ($\sqrt{1-\left(\frac{v}{c}\right)^2}$, where v = speed of Earth in orbit and c = speed of light), or, perhaps, the dimensions of objects transverse to motion through the aether were expanded by an appropriate fraction.

"I would suggest that almost the only hypothesis that could reconcile this opposition [of aberration and the M-M experiment] is that the lengths of material bodies changes according as they are moving through the ether or across it, by an amount depending on the square of the ratio of their velocities to that of light." (*Science* **13**, p. 390, (1889), quoted in *Am. J. Phys.* **69**, p. 1048, (2001).)

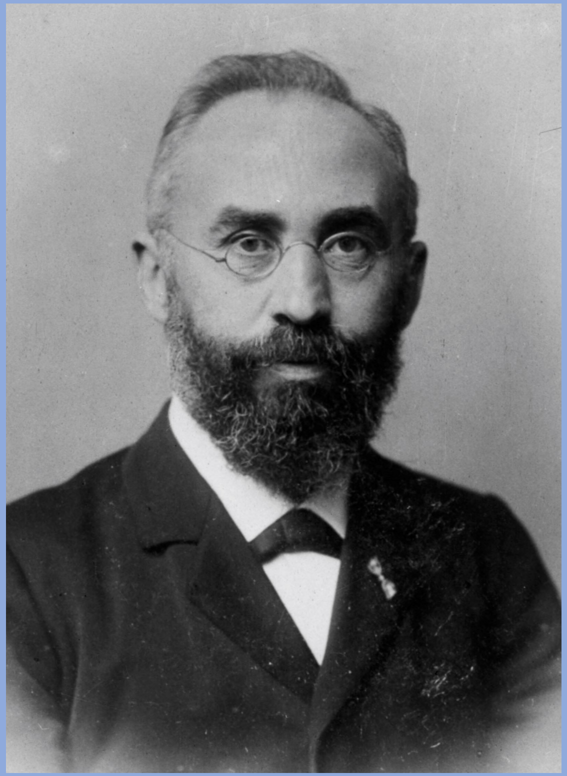
Initial Efforts at Reconciliation (2)

Lorentz Transformations

From 1892 to 1904, Lorentz developed a theory of electromagnetic interactions that indicated moving systems do appear to shrink relative to stationary observers by the factor $1/\sqrt{1-\left(\frac{v}{c}\right)^2}$, and "local time" in the moving system appears to slow by the inverse factor compared to "real time" in the stationary system.

(*The Principle of Relativity*, 1923, Dover Publications, New York, p. 13.)

Hendrik Lorentz (1853-1928)



Initial Efforts at Reconciliation (3)

Electromagnetic Relativity

In 1905 Henri Poincaré showed that Maxwell's equations are the same for reference frames related by what he called "Lorentz transformations." He maintained Lorentz's distinction between "real time" in the stationary system and "local time" in the moving system.

Henri Poincaré (1854-1912)

Albert Einstein’s Special Relativity (1)

In 1905, in his paper “On the Electrodynamics of Moving Bodies” Einstein independently derived the Lorentz transformations from two simple assumptions:

Albert Einstein's Special Relativity (2)

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(1) "the Principle of Relativity," ***i. e.***

All physics laws remain the same in each inertial reference frame,

Albert Einstein's Special Relativity (3)

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(1) "the Principle of Relativity," *i. e.*
All physics laws remain the same in each inertial reference frame, and

(2) "light is always propagated in empty space with a definite velocity *c* which is independent of the state of motion of the emitting body.", *i. e.* Maxwell's equations, which entail the speed *c*, are laws of physics.

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and

(2) "light is always propagated in empty space with a definite velocity *c* which is independent of the state of motion of the emitting body.",
i. e. Maxwell's equations, which entail the speed *c,* are laws of physics.

Consequently,
"[t]he introduction of a 'luminiferous ether' will prove superfluous."

(*The Principle of Relativity*, 1923, Dover Publications, New York, pp. 37-48.)

Lorentz Transformations

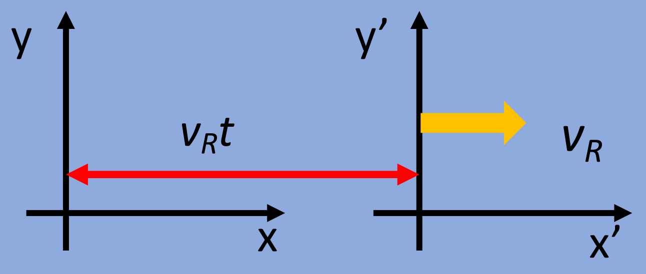
Many AP physics texts give formulas for applying results of Lorentz transformations, *e. g.* length contraction and time dilation, but they don’t give the reasoning that leads to the derivations.

There are many different derivations. The derivation found in *Spacetime Physics* by Edwin Taylor and John Wheeler, is, I think, particularly clear.

We will follow that derivation after a BREAK.

Lorentz Transformations (1)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction. Units are chosen so that $c = 1$, *e. g.* distance in m and time in light-meters, where 1 light-m = (1 m)/c = 3.3 ns, $v_R<1$ is measured as a fraction of c.

If we know the coordinates (time and position) of an event that occurs in either the Lab or Rocket frame, we want transformation equations that allow us to compute the coordinates of that event as viewed in the other frame.

Lorentz Transformations (2)

Starting from the relativity principle, we can derive three corollaries.

Corollary 1: The speed of light in vacuum (*c*) is the same in all inertial reference frames.

If we accept Maxwell's laws of electromagnetism as laws of physics, and they predict the speed of light in vacuum. Then, by the relativity principle, *c* is the same in all inertial frames.

Example: The speed of a light beam emitted by a rocket traveling at $v_R = 0.5\ c$ would be *c* when measured in the Rocket frame and *c* when measured in the Lab frame.

This takes some getting used to before it becomes intuitive.

Lorentz Transformations (3)

Corollary 2: The values of event coordinates transverse (perpendicular) to the direction of relative motion of two reference frames are invariant, *e. g.* Rocket frame moves in the *x*-direction of the Lab frame, $y = y'$ and $z = z'$.

Reductio ad absurdum argument:
Suppose transverse coordinates were not invariant but shrunk in the moving system. Consider a train car with wheels separated by distance W in its rest frame at rest on rails separated by distance R in the rail rest frame.

Lorentz Transformations (4)

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Reductio ad absurdum argument:

Suppose transverse coordinates were not invariant but shrunk in the moving system. Consider a train car with wheels separated by distance W in its rest frame at rest on rails separated by distance R in the rail rest frame. When the train started moving at sufficient speed relative to the rails, an observer standing by the rails would see W shrink and the train wheels fall off inside the rails**. An observer standing in the train car would see R shrink and the wheels fall off outside the rails.**

Lorentz Transformations (6)

Corollary 2: The values of event coordinates transverse (perpendicular) to the direction of relative motion of two reference frames are invariant, *e. g.* Rocket frame moves in the x-direction of the Lab frame, $y = y'$ and $z = z'$.

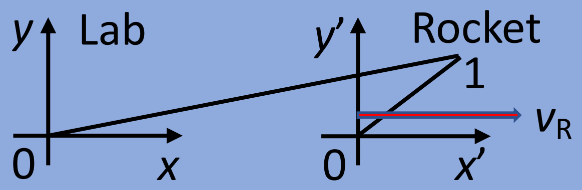
Reductio ad absurdum argument:

Suppose transverse coordinates were not invariant but shrunk in the moving system. Consider a train car with wheels separated by distance W in its rest frame at rest on rails separated by distance R in the rail rest frame. When the train started moving at sufficient speed relative to the rails, an observer standing by the rails would see W shrink and the train wheels fall off inside the rails. An observer standing in the train car would see R shrink and the wheels fall off outside the rails. **Contradictory results prove the original premise was incorrect. Therefore, transvers coordinates are invariant.**

Lorentz Transformations (7)

Corollary 3: The spacetime interval between events is invariant,
i. e. $t^2 - x^2 = t'^2 - x'^2 = \tau^2$ = (proper time)^2 is invariant.

Consider a light flash emitted from the Rocket frame origin at Event 0 $(t',x',y',z')=(t,x,y,z)=(0,0,0,0)$ and received at Event 1 $(t'_1,x'_1,y'_1,0)$, $(t_1,x_1,y_1,0)$. The light beam paths in the Lab and Rocket frames are drawn.



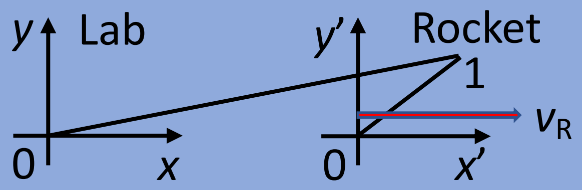
Origins and axes of the frames coincide at Event 0.
Rocket frame moves in x-direction at speed v_R.

Since $c = 1$ in both frames, **(light travel time)^2 = (light path length)^2** in both.

Lorentz Transformations (8)

<u>Corollary 3</u>: The spacetime interval between events is <u>invariant</u>,
e. g. $t^2 - x^2 = t'^2 - x'^2 = \tau^2 =$ (proper time)^2 is invariant.

Consider a light flash emitted from the Rocket frame origin at Event 0 $(t',x',y',z')=(t,x,y,z)=(0,0,0,0)$ and received at Event 1 $(t'_1,x'_1,y'_1,0)$, $(t_1,x_1,y_1,0)$. The light beam paths in the Lab and Rocket frames are drawn.



Origins and axes of the frames coincide at Event 0. Rocket frame moves in x-direction at speed v_R.

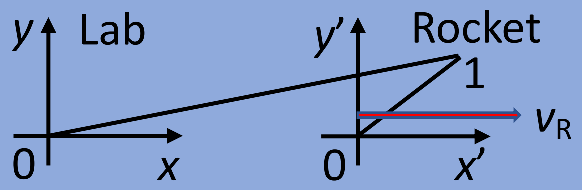
Since $c = 1$ in both frames, (light travel time)^2 = (light path length)^2 in both.

Then $t_1{}^2 = x_1{}^2 + y_1{}^2$ and $t'_1{}^2 = x'_1{}^2 + y'_1{}^2$. So, $t_1{}^2 - x_1{}^2 = y_1{}^2$ and $t'_1{}^2 - x'_1{}^2 = y'_1{}^2$.

Lorentz Transformations (9)

<u>Corollary 3</u>: The spacetime interval between events is <u>invariant</u>,
e. g. $t^2 - x^2 = t'^2 - x'^2 = \tau^2 =$ (proper time)^2 is invariant.

Consider a light flash emitted from the Rocket frame origin at Event 0 $(t',x',y',z')=(t,x,y,z)=(0,0,0,0)$ and received at Event 1 $(t'_1,x'_1,y'_1,0)$, $(t_1,x_1,y_1,0)$. The light beam paths in the Lab and Rocket frames are drawn.



Origins and axes of the frames coincide at Event 0.
Rocket frame moves in *x*-direction at speed v_R.

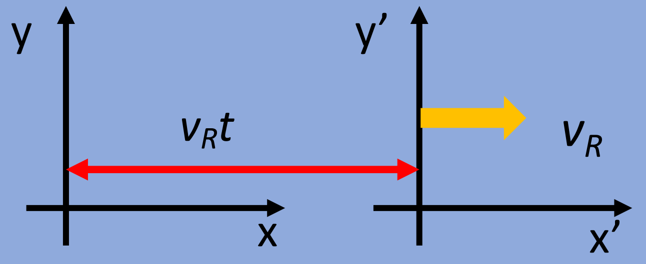
Since $c = 1$ in both frames, (light travel time)^2 = (light path length)^2 in both.

Then $t_1^2 = x_1^2 + y_1^2$ and $t'^2_1 = x'^2_1 + y'^2_1$. So, $t_1^2 - x_1^2 = y_1^2$ and $t'^2_1 - x'^2_1 = y'^2_1$.

But $y_1^2 = y'^2_1$ (invariant). **Therefore, $t_1^2 - x_1^2 = t'^2_1 - x'^2_1$ = invariant.**

Lorentz Transformations (10)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



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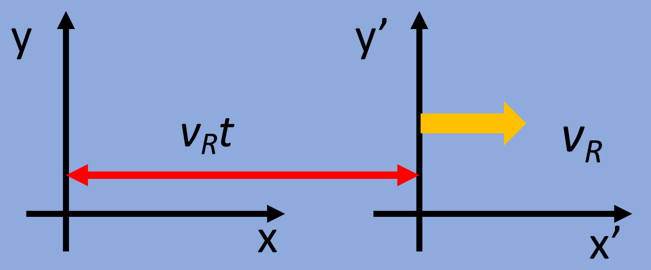
Lorentz Transformations (11)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.

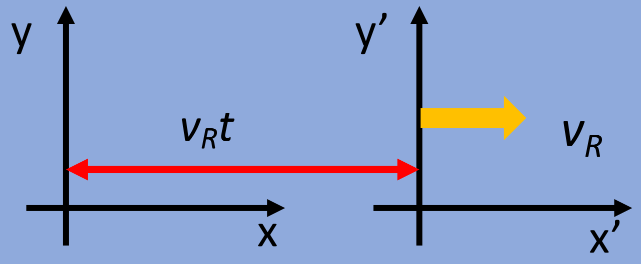


Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction.
Units are chosen so that $c = 1$,
e. g. distance in m and time in light-meters, where 1 light-m = (1 m)/c = 3.3 ns, $v_R<1$ is measured as a fraction of c.

<u>Assume the transformation equations are linear</u> of the form $x = ax' + bt'$ and $t = ex' + ft'$ for one-to-one correspondence between points in the reference frames.

Lorentz Transformations (12)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



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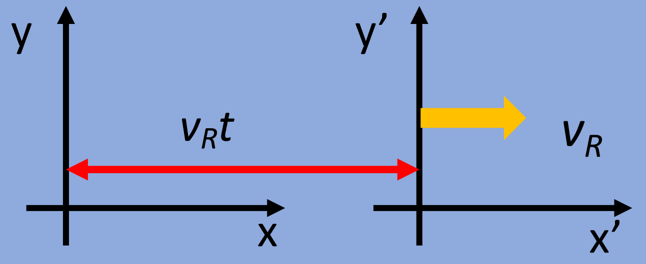
<u>Assume the transformation equations are linear</u> of the form $x = ax' + bt'$ and $t = ex' + ft'$ for one-to-one correspondence between points in the reference frames.

Coefficients *a*, *b*, *e*, *f* may depend on v_R but not on coordinates *x'* or *t'*.

Lorentz Transformations (13)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



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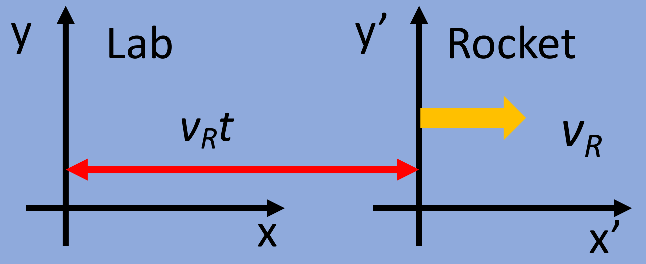
Assume the transformation equations are linear of the form
$x = ax' + bt'$ and $t = ex' + ft'$ for one-to-one correspondence between points in the reference frames.
Coefficients a, b, e, f may depend on v_R but not on coordinates x' or t'.
We need 4 equations to solve for the 4 unknown coefficients.

Lorentz Transformations (14)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction.
Units are chosen so that $c = 1$.

$x = ax' + bt'$ and $t = ex' + ft'$

Consider a light flash emitted from the Lab frame origin in +x-direction at $(t,x,y,z) = (t',x',y',z') = (0,0,0,0)$ and received at Event A $(t_A,x_A,0,0)$, $(t'_A,x'_A,0,0)$.

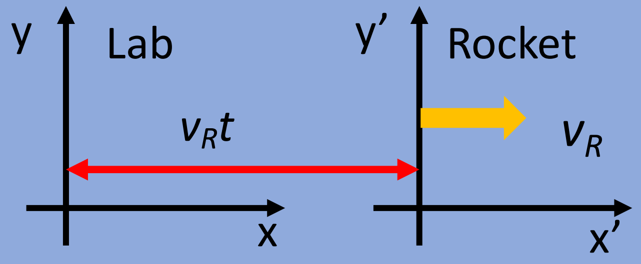
For a light flash, $x_A = t_A$.   Thus, $ax'_A + bt'_A = ex'_A + ft'_A$.

Since $x'_A = t'_A$, we have $at'_A + bt'_A = et'_A + ft'_A$.

Dividing by t'_A gives us $a + b = e + f$.   (Eqn. 1)

Lorentz Transformations (15)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction.
Units are chosen so that $c = 1$.
$x = ax' + bt'$ and $t = ex' + ft'$

Consider a light flash emitted from the Lab frame origin in -x-direction at $(t,x,y,z) = (t',x',y',z') = (0,0,0,0)$ and received at Event B $(t_B,x_B,0,0)$, $(t'_B,x'_B,0,0)$.
For a light flash, $-x_B = t_B$. Thus, $-(ax'_B + bt'_B) = ex'_B + ft'_B$.

Since $-x'_B = t'_B$, we have $at'_B - bt'_B = -et'_B + ft'_B$.

Dividing by t'_B gives us $a - b = -e + f$. (Eqn. 2)

Lorentz Transformations (16)

(Eqn. 1) + (Eqn. 2):

$$\begin{aligned} a + b &= e + f \\ \underline{a - b} &\underline{= -e + f} \\ 2a &= 2f \end{aligned} \quad \text{or} \quad a = f \quad \text{(Eqn. 3)}$$

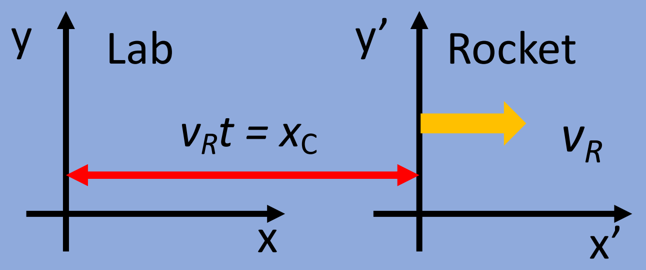
(Eqn. 1) – (Eqn. 2):

$$\begin{aligned} a + b &= e + f \\ \underline{-[(a - b)} &\underline{= (-e + f)\}} \\ 2b &= 2e \end{aligned} \quad \text{or} \quad b = e \quad \text{(Eqn. 4)}$$

We are half-way there!

Lorentz Transformations (17)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction.

Units are chosen so that $c = 1$.

$x = ax' + bt'$ and $t = ex' + ft'$

The origin of the Rocket frame arrives at position x_C at time t_C for Event C.
Event C coordinates are $(t_C, x_C, 0, 0)$ and $(t'_C, 0, 0, 0)$. The Rocket origin is $x' = 0$.

Since $x_C = v_R t_C$, we have $ax'_C + bt'_C = v_R(ex'_C + ft'_C)$.

With $x'_C = 0$, we have $bt'_C = v_R\, ft'_C$ or $b = v_R\, f$ (Eqn. 5)

Lorentz Transformations (18)

Finally, recall the interval invariant: $t'^2 - x'^2 = t^2 - x^2$.

$$t'^2 - x'^2 = (ex' + ft')^2 - (ax' + bt')^2$$

Substitute relations from Eqn. 3, 4, and 5:

$$t'^2 - x'^2 = (v_R f x' + ft')^2 - (fx' + v_R f t')^2$$

After doing some algebra, we get:

$$t'^2 - x'^2 = f^2(1 - v_R^2)(t'^2 - x'^2)$$

Thus, $$f^2(1 - v_R^2) = 1$$

Finally, $f = a = (1 - v_R^2)^{-1/2}$, and $b = e = v_R(1 - v_R^2)^{-1/2}$.

Lorentz Transformations (19)

Let $\gamma = (1 - v_R^2)^{-1/2}$ Note $\gamma \geq 1$.

Lorentz transformations are then

$$t = \gamma(v_R x' + t') \quad x = \gamma(x' + v_R t') \quad y = y' \quad z = z' .$$

Inverse transformations are obtained by exchanging prime and unprime coordinates and reversing the sign of v_R.

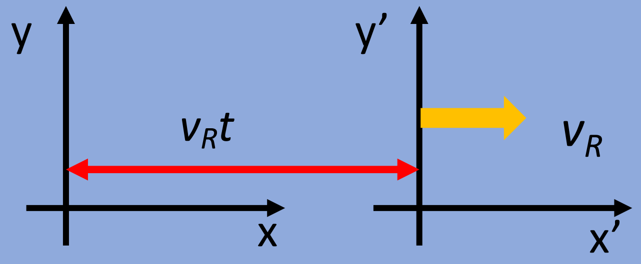
$$t' = \gamma(-v_R x + t) \quad x' = \gamma(x - v_R t) \quad y' = y \quad z' = z .$$

For <u>conventional units</u>, introduce powers of *c* for proper dimensions:

$$\gamma = (1-(v_R/c)^2)^{-1/2} \quad t = \gamma(v_R x'/c^2 + t') \quad x = \gamma(x' + v_R t')$$

Lorentz Transformations (20)

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



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e. g. distance in m and time in light-meters,
where 1 light-m = (1 m)/c = 3.3 ns, $v_R<1$ is measured as a fraction of c.

Let $\gamma = (1 - v_R^2)^{-1/2}$ Note $\gamma \geq 1$.

$t = \gamma(v_R x' + t')$ $\quad x = \gamma(x' + v_R t')$ $\quad y = y'$ $\quad z = z'$

$t' = \gamma(-v_R x + t)$ $\quad x' = \gamma(x - v_R t)$ $\quad y' = y$ $\quad z' = z$

$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2$ $\quad \Delta\tau$ = *invariant proper time between events*

Lorentz Transformations (21)

Let $\gamma = (1 - v_R^2)^{-1/2}$ **Note $\boldsymbol{\gamma} \geq \mathbf{1}$.**

Time dilation:

A clock at the origin of the Rocket frame ($x' = 0$) has time t' between tics. As measured by clocks in the Lab frame, the time between tics of the Rocket clock is $t = \gamma t' > t'$. The Rocket clock appears to run slow.

Length contraction:

A stick at rest on the x'-axis in the Rocket frame has one end at $x'_0 = 0$ and the other at $x'_1 = L$. When the positions of both ends are measured in the Lab frame at $t = 0$, one end is at $x_0 = 0$ and the other end at $x'_1 = \gamma x_1$, so that $x_1 = x'_1/\gamma = L/\gamma < L$. The stick appears shorter in the Lab.

Galilean Transformations

When v_R<<1 (c = 1), $\gamma \cong 1$ and Lorentz transformations are approximated by

$$t = 1(v_R x' + t') \text{ and } x = 1(x' + v_R t')$$

or

$$t = t' \qquad \text{and } x = x' + v_R t'.$$

For most measurements t' (light-m) is vastly greater than x' (m). Since v_R<<1, we have $v_R x'$<<<t'. Thus, event times are approximately the same in both frames.

Thus, Galilean transformations approximate the Lorentz transformations for low velocities of relative motion compared to c.

Time Dilation

The 1962 film "Time Dilation: An Experiment with Mu-Mesons" (35:40) (https:www.youtube.com/watch?v=rbzt8gDSYIM) was made when muons were still called mu-mesons.

The experiment with cosmic ray muons presents a dramatic demonstration of extreme time dilation, as viewed in the Earth frame, and extreme length contraction, as viewed in the muon frame.

Gravitational Influence on Clocks

Einstein proposed his General Relativity (GR) in 1915 to account for accelerating reference frames. Karl Schwarzschild in 1916 found a solution to Einstein's field equations for the case of a spherically symmetric, non-spinning, non-electrically charged mass. The solution relates measurements of space and time coordinates of events that occur in space at a distance *r* from the center of the mass *M*. The time between tics (t_r) of a clock at distance *r* compared to the time between tics (t_∞) of a clock a great distance ($r = \infty$) from the mass is given by $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$, where *G*, *M*, *r*, *c*, and both *t* values are in SI units.

Black Holes

Note that the expression $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$ goes to zero when $r_{EH} = \frac{2GM}{c^2}$. The r_{EH} value is the radius of the Event Horizon around a Schwarzschild black hole of mass M. Calculate r_{EH} for the following:

Earth Mass $M_E = 6 \times 10^{24}$ kg $r_{EEH} = ?$

Sun Mass $M_S = 2 \times 10^{30}$ kg $= 3.3 \times 10^5\ M_E$ $r_{SEH} = ?$

SgrA* Mass $M_{SgrA*} = 4 \times 10^6\ M_S$ $r_{SgrA*EH} = ?$

Black Holes

Note that the expression $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$ goes to zero when $r_{EH} = \frac{2GM}{c^2}$. The r_{EH} value is the radius of the Event Horizon around a Schwarzschild black hole of mass *M*. Calculate r_{EH} for the following:

Earth Mass $M_E = 6 \times 10^{24}$ kg $r_{EEH} = 9 \times 10^{-3}$ m = 1 cm

Sun Mass $M_S = 2 \times 10^{30}$ kg = $3.3 \times 10^5\ M_E$ $r_{SEH} = 2.9 \times 10^3$ m = 2.9 km

SgrA* Mass $M_{SgrA*} = 4 \times 10^6\ M_S$ $r_{SgrA*EH} = 1.2 \times 10^{10}$ m = 0.1 AU

Navigation Programs

The 2019 video "How Does GPS Actually work and Why Many GPS Devices are About to Stop Working" (14:35) gives a brief history of the navigation system and how it works. It omits reference to relativistic corrections.
(https://www.youtube.com/watch?v=CnwAJrDikgU)

Google Maps and GPS

- Google Maps combines a set of maps, images, and other information about places on Earth with current location information determined by your phone acting as a GPS receiver.
- The Global Positioning System (GPS) is a constellation of at least 24 satellites (currently 31, including spares). The satellites, maintained by the U. S. Space Force, have been placed in orbits at an altitude of about 20,200 km above Earth or 26,580 km from Earth's center. The orbital period is 12 hours, which you can calculate from Kepler's law of periods or Newton's gravitation law. Satellite orbits are inclined at 55 degrees to Earth's equator so that at least four are above the horizon at any one time at almost any place on Earth. The first group of satellites, which are no longer operational, was launched in the 1990s. New groups are regularly launched with updated features. The U. S. Space Force continually tracks the satellites with ground-based radars to verify their positions, synchronize their clocks, and update their onboard ephemeris equations and almanac data that are used to broadcast the satellite positions

GPS and Relativity

The following analysis is adapted from the Wikipedia article "Error analysis for the Global Positioning System"
https://en.wikipedia.org/wiki/Error_analysis_for_the_Global_Positioning_System#:~:text=In%20the%20context%20of%20GPS,body)%20appear%20to%20tick%20slower.

Kinetic Time Dilation (1)

The factor by which clocks in the GPS satellites tick slower, due to satellite velocity, than clocks stationary on Earth is determined using the Lorentz transformation. Time measured by an object with velocity *v* compared to a stationary object is given by (the inverse of) the Lorentz factor, γ:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} = \sqrt{1 - v^2/c^2}$$

For small values of *v/c,* this ratio, by the binomial approximation, is approximately:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} \cong 1 - \frac{v^2}{2c^2}$$

The GPS satellites, with orbit radius about 26580 km and 12-hour orbit period, move at about 3870 m/s relative to Earth's center.

Kinetic Time Dilation (2)

We thus calculate:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} \cong 1 - \frac{v^2}{2c^2} \cong 1 - \frac{(3870\ \mathrm{m/s})^2}{2(2.998 \times 10^8\ \mathrm{m/s})^2} \cong 1 - 8.332 \times 10^{-11}$$

This value of -8.332×10^{-11} represents the difference in the rate by which the GPS satellite clocks tick slower than Earth-stationary clocks. That rate difference multiplied by the number of nanoseconds in a day yields the nanoseconds per day lost by GPS clocks relative to Earth clocks due to satellite speed:

$(-8.332 \times 10^{-11})(86400\ \mathrm{s/day})(10^9\ \mathrm{ns/s}) \cong -7210\ \mathrm{ns/day}$

In other words, the GPS satellite clocks are slower than clocks at rest on Earth by 7210 nanoseconds per day due to GPS satellite clock velocity.

Kinetic Time Dilation (3)

Note that this speed of 3870 m/s as measured relative to Earth's center rather than its surface where the GPS receivers (and users) are. This is because Earth's equipotential makes net time dilation equal across its geodesic surface. That is, the combination of Special and General effects makes the net time dilation at the equator equal to that of the poles, which in turn are at rest relative to the center. Hence, we use the center as a reference point to represent the entire surface.

Gravitational Time Dilation (1)

The Schwarzschild metric gives the relation between time kept by a stationary clock at distance *r* from a spherical mass *M* compared to the time kept by a stationary clock far away $(r \cong \infty)$ from the mass.

$$\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$$

where t_r is the time passed between events, *e. g.* clock tics, measured by a clock at a distance r from the center of the Earth and t_∞ is the time passed between the events as measured by a far-away observer. *G* is the Newtonian gravitation constant, and *M* is the Earth mass for the case of clocks on Earth and clocks in GPS satellites.

For small values of $GM/(c^2 r)$ this ratio is approximately:

$$\frac{t_r}{t_\infty} \cong 1 - \frac{GM}{c^2 r}$$

Gravitational Time Dilation (2)

The clocks in the GPS satellites orbiting in a weaker gravitational field at a distance of about 4.2 Earth radii from Earth's center tic faster than identical clocks on Earth by a ratio: $t_{\mathrm{GPS}}/t_{\mathrm{Earth}}$:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{t_{GPS}/t_{\infty}}{t_{Earth}/t_{\infty}} \cong \left(1 - \frac{GM}{c^2 r_{GPS}}\right)\left(1 - \frac{GM}{c^2 r_{Earth}}\right)^{-1} \cong \left(1 - \frac{GM}{c^2 r_{GPS}}\right)\left(1 + \frac{GM}{c^2 r_{Earth}}\right)$$

$$\frac{t_{GPS}}{t_{Earth}} \cong 1 + \left(\frac{GM}{c^2 r_{Earth}} - \frac{GM}{c^2 r_{GPS}}\right) \cong 1 + 5.307 \text{ x } 10^{-10},$$

for r_{Earth} = 6,357,000 m, r_{GPS} = 26,541,000 m, Earth $M = 5.974\times10^{24}$ kg,

$G = 6.674\times10^{-11}$ m^3 kg^{-1}s^{-2}, and $c = 2.998\times10^8$ m/s.

Gravitational Time Dilation (3)

The value 5.307 x 10^{-10} represents the fraction by which the clocks at GPS satellite's altitude tick faster than identical clocks on the surface of Earth. This fraction multiplied by the number of nanoseconds in a day yields the nanoseconds per day gained by GPS clocks relative to Earth clocks due to the difference in the local gravitational field:

$$(+5.307 \times 10^{-10})(86400 \text{ s/day})(10^{9} \text{ ns/s}) \cong +45850 \text{ ns/day}$$

Thus, the satellites' clocks gain 45850 nanoseconds a day due to gravitational time dilation.

Combined Time Dilation Effects

These effects are added together to give (rounded to 10 ns):

45850 – 7210 = 38640 ns/day

Hence, the satellites' clocks gain approximately 38,640 nanoseconds a day or 38.6 μs per day due to relativistic effects in total.

To compensate for this gain, a GPS clock's frequency needs to be slowed by the fraction:

$5.307 \times 10^{-10} - 8.349 \times 10^{-11} = 4.472 \times 10^{-10}$

This fraction is subtracted from 1 and multiplied by the pre-adjusted clock frequency of 10.23 MHz:

$(1 - 4.472 \times 10^{-10}) \times 10.23 = 10.22999999543$

In other words, we need to slow the clocks down from 10.23 MHz to 10.22999999543 MHz to negate both time dilation effects.

Sources of User Equivalent Range Errors

Source	Effect (m)
Signal arrival C/A	±3
Signal arrival P(Y)	±0.3
Ionospheric effects	±5
Ephemeris errors	±2.5
Satellite clock errors	±2
Multipath distortion	±1
Tropospheric effects	±0.5
3σR C/A (code)	±6.7
3σR P(Y) (code)	±6.0

Conclusion

Every time we use a GPS receiver that accurately tell us where we are, we verify predictions of Einstein’s special and general relativity theories.

Thank you, Albert, for helping us find our way home!